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ON RINGS SATISFYING THE IDENTITY

$$(x+x^2+\cdots+x^n)^{(n)}=0$$

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Throughout the present paper, R will represent a ring with center C . Let N be the set of nilpotents in R , N^* the subset of N consisting of all a with $a^2=0$, E the set of idempotents in R , and D the commutator ideal of R . For $x \in R$, we define inductively $x^{(1)}=x$, $x^{(k)}=x^{(k-1)} \circ x$, where $x \circ y = x+y+xy$. We may write formally $x^{(k)}=(1+x)^k-1$.

Let n be a positive integer, and consider the following properties:

- (i)_n $(x+x^2+\cdots+x^n)^{(n)}=0$ for all $x \in R$.
 - (ii) N is commutative.
 - (ii)* N^* is commutative.
 - (iii) $[[a,x],x]=0$ for all $a \in N$ and $x \in R$.
 - (iv) If $a \in N$, $x \in R$ and $[a,x]^2=0$ then $[a,x] \in C$.
 - (*) For any $x, y \in R$, $(x+xy) \circ (y+yx)=0$ if and only if $x=y$.
- If R has 1, then (i)_n becomes
- (i)'_n $(1+x+x^2+\cdots+x^n)^n=1$ for all $x \in R$.

The present objective is to prove the following theorems.

Theorem 1. *Let R be a left s -unital ring satisfying (i)_n. If R is normal (i.e., E is central) then N is a nil ideal and R/N is commutative.*

Theorem 2. *Let R be a left s -unital ring satisfying (i)_n.*

(1) *If R satisfies (ii)*, then N is a nil ideal and R/N is a commutative regular ring.*

(2) *If R satisfies (ii), then N is a commutative nil ideal and R/N is a commutative regular ring.*

(3) *If R satisfies (ii) and (iii) (or (iv)), then R is commutative and R/N is a regular ring.*

Theorem 3. *Let R be a left s -unital ring satisfying (i)_{2*}. Then N is a nil ideal and $R=R_1 \oplus R_2$, where R_1 is either 0 or a commutative regular ring of odd characteristic, $R_2 \supseteq N$ and R_2/N is a Boolean ring.*

Theorem 4. *If R is a normal, left s -unital ring satisfying (i)_{2*} and*

(ii), then R is commutative and $R = R_1 \oplus R_2$, where R_1 is either 0 or a regular ring of odd characteristic, $R_2 \supseteq N$ and R_2/N is a Boolean ring.

Theorem 5 (cf. [4, Theorem 2]). A left s -unital ring R satisfies $(*)$ if and only if (a) R is commutative and R/N is a Boolean ring and (b) $a^{(2)} = 0$ for all $a \in N$.

We start with the following lemma.

Lemma 1. Suppose that R satisfies $(i)_n$ and $p^a R = 0$, where p is a prime. Then there exists a positive integer m such that $x^m = x^{2^m}$ for all $x \in R$.

Proof. Let $y = x + x^2 + \cdots + x^n$, and $n = p^\beta t$, where $\beta \geq 0$ and $(p, t) = 1$. Then there exists $u(\lambda) \in \mathbb{Z}[\lambda]$ such that $y^{(t)} = (1+y)^t - 1 = tx + x^2 u(x)$. Noting here that $(pR)^a = 0$, we can easily see that $(tx + x^2 u(x))^{p^\beta a} = 0$. Because $(t, p) = 1$, we readily obtain $x^{p^\beta a} - x^{p^\beta a+1} f(x) = 0$ with some $f(\lambda) \in \mathbb{Z}[\lambda]$. Now, by making use of the argument employed in the proof of [1, Lemma], we can find a positive integer m such that $x^m = x^{2^m}$ for all $x \in R$.

Remark 1. (1) Recently, Komatsu [5] proved that a ring R with 1 satisfies a polynomial identity $x^{2^m} - x^m = 0$ for some positive integer m if and only if the addition of R is equationally definable in terms of the multiplication and the successor operation.

(2) Let p be a prime. If $(1+n)^n \equiv 1 \pmod{p}$ and $p-1 \mid n$, then $\text{GF}(p)$ satisfies $(i)_n$. For instance, $\text{GF}(3)$ satisfies $(i)_4$.

(3) Let R be a left s -unital ring satisfying $(i)_n$. Let x be an arbitrary non-zero element of R , and choose $e \in R$ such that $ex = x$. Then, by $(i)_n$,

$$0 = \{(1+e+e^2+\cdots+e^n)^n-1\}x = \{(n+1)^n-1\}x.$$

This means that the characteristic of R is non-zero. Furthermore, n has to be even. In fact, if n is odd, then

$$0 = \{(1-e+e^2-\cdots-e^n)^n-1\}x = -x,$$

which is a contradiction. In particular, if p is a prime and $n = p^k$ then $p = 2$.

Proof of Theorem 1. By Remark 1 (3), the characteristic of R is non-zero. Obviously, the hypothesis $(i)_n$ and the normality are inherited by subrings, so we may assume that the additive group of R is a p -group, and

therefore $p^a R = 0$ for some prime p . Then, by Lemma 1, there exists a positive integer m such that $x^m = x^{2^m}$ for all $x \in R$, and so R satisfies the polynomial identity $[x^m, y] = 0$. Since $[E_{11}, E_{11} + E_{12}] = E_{12} \neq 0$ in $(GF(q))_2$ (q a prime), D is a nil ideal by [3, Proposition 2].

Remark 2. For $e \in E$, the following are equivalent :

- 1) $e \in C$.
- 2) $[e, a] = 0$ for all $a \in N^*$.
- 3) $[e, [e, a]] = 0$ for all $a \in N^*$.
- 4) $[(e+a)e, e(e+a)] = 0$ for all $a \in N^*$.
- 5) $[e, f] = 0$ for all $f \in E$.

In fact, it is clear that 1) implies 2)–5), and 2) does 3). In order to see that each of 3)–5) implies 1), given $x \in R$, we set $a = ex(1-e) \in N^*$. It is easy to see that if 3) or 4) is satisfied then $a = 0$, i.e., $ex = exe$. Similarly, we can see that $xe = exe$. Finally, if 5) is satisfied, then $e+a \in E$ and $e+a = e(e+a) = (e+a)e = e$, whence it follows again $a = 0$, i.e., $ex = exe$; similarly $xe = exe$. In particular, R is normal if and only if E is commutative, and (iii) implies the normality of R . Furthermore, if N^* is central then R is normal and N coincides with the prime radical of R . In fact, if $a^n = 0$ and $a^{n-k}(Ra)^{k+1} = 0$ then $\{a^{n-k-1}(Ra)^{k+1}\}^2 \subseteq a^{n-k-1}Ra^{n-k}(Ra)^{k+1} = 0$. Hence $a^{n-k-1}(Ra)^{k+1} \subseteq C$, and so $a^{n-k-1}(Ra)^{k+2} = Ra^{n-k}(Ra)^{k+1} = 0$. We get eventually $(Ra)^{n+1} = 0$.

In advance of proving Theorem 2, we state the next lemma.

Lemma 2. (1) If R satisfies (ii)* then R/P is normal, where P is the prime radical of R .

(2) If a π -regular ring R satisfies (ii)*, then N coincides with the Jacobson radical of R and R/N is strongly regular.

Proof. (1) Let \bar{e} be an arbitrary idempotent of $\bar{R} = R/P$. Since P is a nil ideal, we may assume from the beginning that e is in E . By hypothesis, $eR(1-e) \cdot (1-e)Re = (1-e)Re \cdot eR(1-e)$, and so $eR(1-e)Re = 0$. Hence $\bar{e}\bar{R}(1-\bar{e})\bar{R}\bar{e} = 0$. By the semiprimeness of \bar{R} , we get $\bar{e}\bar{R}(1-\bar{e}) = 0$, and therefore $\bar{e}\bar{x} = \bar{e}\bar{x}\bar{e}$ for all $x \in R$. Furthermore, $\bar{e}\bar{R}(1-\bar{e})\bar{R} = 0$ yields $(1-\bar{e})\bar{R}\bar{e} = 0$, and therefore $\bar{x}\bar{e} = \bar{e}\bar{x}\bar{e}$ for all $x \in R$. Thus, we have seen that \bar{e} is central.

(2) Let J be the Jacobson radical of the π -regular ring R . Obviously, J/P is a nil ideal of R/P . Since $R/J \simeq (R/P)/(J/P)$ and R/P is normal by (1), it is easy to see that R/J is normal and any nilpotent element of

R/J generates a nil right ideal. Hence, R/J is reduced and N coincides with J . The reduced π -regular ring R/N is strongly regular (see, e.g. [2]).

Proof of Theorem 2. (1) As in the proof of Theorem 1, we may assume that $p^a R = 0$ with some prime p . Then, by Lemma 1, there exists a positive integer m such that $x^m = x^{2m}$ for all $x \in R$. Hence, by Lemma 2 (2), N coincides with the Jacobson radical of R and R/N satisfies the identity $x - x^{m+1} = 0$. By Jacobson's commutativity theorem, R/N is commutative.

(2) This is clear by (1).

(3) By the proof of (1) and [6, Theorem 1], R is commutative.

Lemma 3. *If R satisfies $(i)_{2^k}$ and $2^a R = 0$, then N is a nil ideal and R/N is a Boolean ring.*

Proof. Let $n = 2^k$. Obviously, $y = x + x^2 + \cdots + x^n \in N$, and so $x - x^{n+1} = y - xy \in N$. Then, noting that $x(1-x)^n - (x - x^{n+1}) \in 2R \subseteq N$, we readily see that $x - x^2 = x(1-x) \in N$. Now, we claim that for any prime q , $(\text{GF}(q))_2$ fails to satisfy $(i)'_n$. If $q \neq 2$ then $x = E_{12}$ does not satisfy $(i)'_n$. On the other hand, if $(\text{GF}(2))_2$ satisfies $(i)'_n$ then, as we have seen just above, $x - x^2$ is nilpotent for all $x \in (\text{GF}(2))_2$. But this is not true for $x = E_{11} + E_{12} + E_{21}$. Thus, by [3, Proposition 2], D is a nil ideal, and so R/N is a Boolean ring.

Proof of Theorem 3. By Remark 1 (3), $hR = 0$ for some positive integer $h = 2^a h'$, $(2, h') = 1$. Then, $R = R_1 \oplus R_2$, where $h'R_1 = 0$ and $2^a R_2 = 0$. If a is an element of R_1 with $a^2 = 0$ then, by $(i)_{2^k}$, we can easily see that $2^k a = 0$, and therefore $a = 0$. This proves that R_1 is a reduced ring and $N \subseteq R_2$. Now, the assertion is an easy combination of Theorem 2 (1) and Lemma 3.

Proof of Theorem 4. In view of Theorem 3, it remains only to prove that R_2 is commutative. Obviously, every element of R_2 is of the form $e + a$, where $e \in E \cap R_2$ and $a \in N$. Since E is central and N is commutative, it is immediate that R_2 is commutative.

Remark 3. As the following example shows, Theorem 4 is not true if we replace 2^k by an arbitrary positive integer: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \text{GF}(4) \right\}.$$

Then $12R = 0$ and R is a normal ring satisfying (i)₁₂ and (ii). But R is not commutative and R/N is not Boolean either.

Proof of Theorem 5. "Only if": Obviously, (*) implies (i)₂. We claim that R is normal. To see this, given $e \in E$ and $x \in R$, we put $a = ex(1-e)$. Since

$$\{(a-e)-(a-e)e\} \circ \{-e-e(a-e)\} = a \circ (-a) = -a^2 = 0,$$

(*) shows that $a-e = -e$, and so $a = 0$, i.e., $ex = exe$. A similar argument gives $xe = exe$, and hence $ex = xe$. It is easy to see that $8R = 0$, and therefore R is a normal, left s -unital ring with $2^3R = 0$ and satisfies (i)₂. We shall show that R satisfies (ii). Let x be an arbitrary element of R , and let e be such that $ex = x$. Since $(x+x^2)^{(2)} = 0 = (-x+x^2)^{(2)}$ by (i)₂, we readily obtain $4(x+x^3) = 0$, i.e., $4x = 4x^3$. Replacing x by $e+x$ in the last, we get

$$4x+4x^2 = 4(e+x)x = 4(e+x)^3x = 4x+4x^2+4x^3-4x^4.$$

Combining this with $4x = 4x^3$, we see that $4x = 4x^4 = 4x^3x = 4x^2$. Since $x^4 = -2x^3 - 3x^2 - 2x$ by (i)₂, we get

$$x^5 = -2x^4 - 3x^3 - 2x^2 = -2(-2x^3 - 3x^2 - 2x) - 3x^3 - 2x^2 = x^3 + 4x^2 - 4x = x^3.$$

This implies that $a^3 = 0$ for all $a \in N$, and therefore

$$a+a^2 = (a+a^2)^{(2)} \circ (-a) = -a, \text{ i.e., } a^{(2)} = 0.$$

Noting that N is a nil ideal by Theorem 1, we get for any $a, b \in N$

$$a \circ b = a \circ (a \circ b)^{(2)} \circ b = b \circ a,$$

which shows that N is commutative. Thus, by Theorem 4, R is commutative and R/N is a Boolean ring.

"If": First, we claim that every quasi-regular element of R is nilpotent. In fact, if a is quasi-regular then the nilpotency of $a+a^2$ yields that of a . Obviously, $(x+x^2)^{(2)} = 0$ for all $x \in R$. Conversely, if $(x+xy) \circ (y+yx) = 0$ then $x+xy$ is nilpotent, and hence $y+xy = (x+xy)^{(2)} \circ (y+yx) = x+xy$, whence it follows that $y = x$.

Remark 4. In order to prove the only if part of Theorem 5, we quoted Theorems 1 and 4. In case R has 1, we can prove it more directly. In fact, $x^5 = x^3$ yields that $u^2 = 1$ for every unit u in R . If u, v are units in R then $uv = (uv)^{-1} = v^{-1}u^{-1} = vu$. In particular, if a and b are in N then $[a, b] = [1+a, 1+b] = 0$, and so N is commutative. Now, let $a \in N$, $x \in R$. Since $(x+x^2)^6 = -8(x+x^2)^3 = 0$ and $(x^2+x^4)^6 = 0$ by (i)₂ and x^4

is a central idempotent, we have $[a, x] = [a, x - x^4] = [a, x + x^2] - [a, x^2 + x^4] = 0$, and hence $N \subseteq C$. Therefore, $x = (x + x^2) - (x^2 + x^4) + x^4 \in C$ for all x in R , and hence R is commutative.

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